

# Endogenous Fertility with Quasilinear Preferences<sup>\*</sup>

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## Abstract

This paper presents a static model with endogenous fertility, in which an individual has quasilinear preferences. In this case, an individual does not always have a child. First, we compare the fertility in the case of homothetic preferences with that of quasilinear preferences. Second, we consider the effects of child policy on fertility. If the government ensures a child allowance, fertility increases even if the government raises tax. This results from a strong income effect.

**Keywords:** Endogenous fertility, Homothetic preferences, Quasilinear preferences, Taxation, Child policy

**JEL Classifications:** H24, J13, J18

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## 1 Introduction

There are considerable literatures concerning endogenous fertility, such as Becker and Barro (1989). In these literatures, it is assumed that the utility function is homothetic. Using homothetic preferences, it is possible to calculate an equilibrium in the dynamic model. Hence, homothetic preferences are used in various economics models. Under this assumption, an individual always has a child even if his/her income is infinitesimally low, except in the case of 0. In addition, when an individual has homothetic preferences, the income elasticity of fertility is positive and constant; i.e., fertility is determined proportional to income. However, these seem harder for explaining an individual's decision for having a child.

In this paper, we present the model where an individual has quasilinear preferences. Because it is difficult to calculate the dynamics of fertility, we consider endogenous fertility problematic in the static model. In this case, we show that an individual does not always have a child when his/her income is extremely low. We demonstrate the income level at which an individual has a child. Moreover, we consider the effects of child policy on fertility.

The remaining paper is organized as follows. Section 2 states the basic model whose utility function is homothetic. Section 3 shows the model where an individual has quasilinear preferences. Section 4 discusses child policy for fertility. The final section concludes this paper.

## 2 The Basic Model

### 2.1 Firm

We consider a perfectly competitive economy. Firms are identical, and factor markets are perfectly competitive. There is a good for individual consumption, which is produced using a linear technology that employs only labor. A production function  $Y$  is defined as follows:

$$Y = AL, \quad A > 1,$$

where  $L$  and  $A$  denote the labor and the productivity of labor, respectively. In addition, we assume that  $A$  is constant. Taking consumption good as the numeraire, we assume the price as one. Then, given the wage rate  $w$ , the profit of a firm,  $\pi$ , is the following.

$$\pi = Y - wL.$$

Solving the profit maximization problem, we get the following first order condition.

$$MP_L = A \equiv w.$$

Therefore, the labor wage is constant over time.

### 2.2 Household

Each individual is endowed with one unit of time and inelastically, supplying one unit of labor to a firm. This individual receives wage  $w$  and divides them among the consumption and the rearing costs of children. We assume that the rearing cost per child is  $m + \beta w$ , which is the sum of consumption  $m$  and opportunity cost  $\beta w$  ( $\beta > 0$ ) for a child<sup>1</sup>. Then, the household's budget constraint is expressed by the following equation:

$$w = c + (m + \beta w)n, \tag{1}$$

where  $c$  and  $n$  represent the consumption and the number of children, respectively.

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1 See Boldrin and Jones (2002) and Fanti and Gori (2009).

An individual cares about the consumption and the number of children and the parent only makes decisions in the economy. Then, the individual's utility function is specified as follows:

$$u = \gamma \log c + (1 - \gamma) \log n, \quad \gamma \in (0, 1), \quad (2)$$

where  $\gamma$  and  $(1 - \gamma)$  denote preference toward consumption and the number of children, respectively<sup>2</sup>. Hence, an individual maximizes equation (2) subject to equation (1).

Solving the utility maximization problem, we get the optimal solutions as follows:

$$n = \frac{(1 - \gamma) w}{(m + \beta w)}, \quad (3)$$

$$c = \gamma w. \quad (4)$$

Then, differentiating equation (3) with respect to  $w$ , we get the first-derivative and second-derivative as follows:

$$\begin{aligned} \frac{dn}{dw} &= \frac{(1 - \gamma) m}{(m + \beta w)^2} > 0, \\ \frac{d^2n}{dw^2} &= -\frac{2\beta(1 - \gamma) m}{(m + \beta w)^3} < 0. \end{aligned}$$

Therefore, fertility increases with an increase in wage, as described by Figure 1. In addition, the income elasticity of a child is expressed by

$$\frac{dn}{dw} \frac{w}{n} = \frac{m}{m + \beta w}.$$

The income elasticity of child is positive, less than 1, and constant.

With homothetic preferences, there is no way of deciding not to have children. Therefore, even if a household income is considerably low, an individual may have some children.

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2 In this section, we suppose the homothetic preferences that are used generally in theoretical analysis.

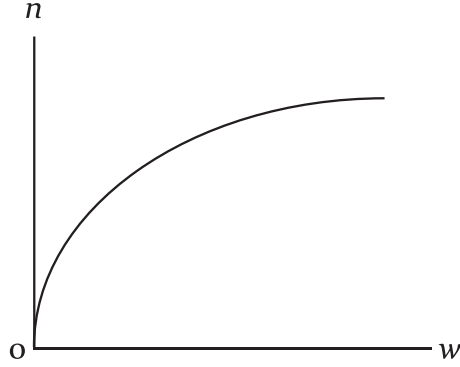


Figure 1. The optimal fertility in the homothetic preferences case

### 3 The Case of Quasilinear Preferences

#### 3.1 The Model

In the previous section, we show endogenous fertility in the case in which an individual has homothetic preferences. In this case, although income is very low (except ), an individual with homothetic preferences has some children. In this section, we present the model in which an individual has quasilinear preferences. First, we provide the model in this case and compare with the fertility in the homothetic preferences case. Second, we consider the effect of child policy on fertility by introducing the government.

Similar to the previous section, an individual cares about the consumption and the number of children. In this section, in place of equation (2), the utility function is specified as follows:

$$u = \gamma \log c + (1 - \gamma) n. \quad (5)$$

Then, an individual utility maximization problem is

$$\begin{aligned} \max_{c,n} u &= \gamma \log c + (1 - \gamma) n, \\ \text{s. t. } w &= c + (m + \beta w) n. \end{aligned}$$

Solving the above problem, the optimal fertility and consumption are as follows:

$$n = \frac{w}{m + \beta w} - \frac{\gamma}{1 - \gamma}, \quad \text{if } w > \frac{\gamma m}{1 - \gamma - \beta \gamma}, \quad (6.a)$$

$$c = \frac{\gamma}{1 - \gamma} (m + \beta w), \quad \text{if } w > \frac{\gamma m}{1 - \gamma - \beta \gamma}, \quad (6.b)$$

$$n = 0, \quad \text{if } w \leq \frac{\gamma m}{1 - \gamma - \beta \gamma}, \quad (7.a)$$

$$c = w, \quad \text{if } w \leq \frac{\gamma m}{1 - \gamma - \beta \gamma}. \quad (7.b)$$

Differentiating equation (6), we have

$$\begin{aligned} \frac{dn}{dw} &= \frac{m}{(m + \beta w)^2} > 0, \\ \frac{d^2n}{dw^2} &= -\frac{2\beta m}{(m + \beta w)^3} < 0. \\ \frac{dn}{dw} &> 0 \end{aligned}$$

where  $n$  is equal to 0 until income reaches  $\frac{\gamma m}{1 - \gamma - \beta \gamma}$ . After the income increases more than  $\frac{\gamma m}{1 - \gamma - \beta \gamma}$ , the value of  $n$  increases in the income. Therefore, fertility is depicted in Figure 2.

In addition, the income elasticity of a child is as follows:

$$\frac{dn}{dw} \frac{w}{n} = \frac{(1 - \gamma) mw}{(m + \beta w) \{ (1 - \gamma - \beta \gamma) w - \gamma m \}}.$$

In this case, the above income elasticity depends on income.

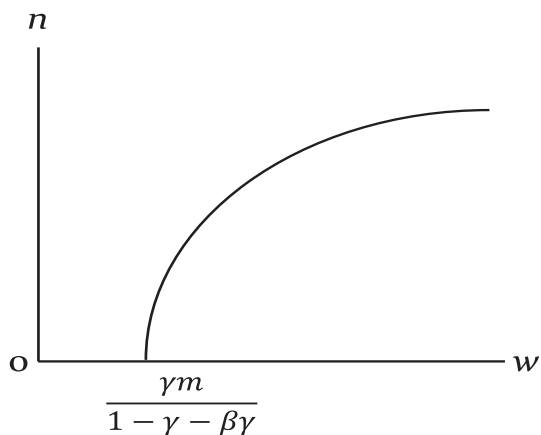


Figure 2. The optimal fertility in the quasilinear preferences case

On the other hand, consumption is monotonically increasing with respect to income. The income consumption curve is shown in Figure 3.

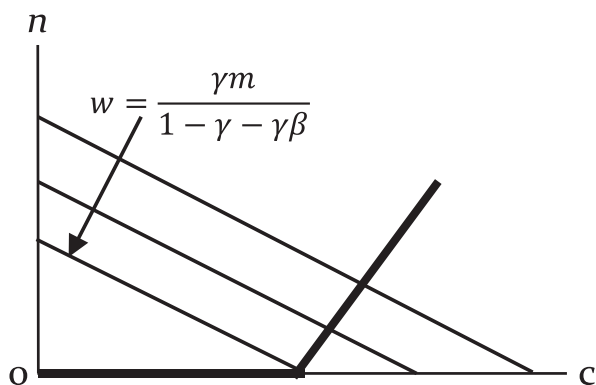


Figure 3. Income consumption curve

## 4 Analyses of Child Policy

### 4.1 The Model

In this section, we consider child policy, in which the government provides child allowance for a household. To finance this child allowance, the government imposes a lump-sum tax,  $\tau$ , on every individual<sup>3</sup>. In addition, we suppose that the size of population is expressed by  $N$  in the economy.

By denoting the child allowance per child by  $z$ , the budget constraint of a household is

$$w - \tau + zn = c + (m + \beta w) n. \quad (8)$$

An individual makes decisions on consumption and fertility so that he/she maximizes utility subject to equation (8). The household problem is as follows:

$$\begin{aligned} \max_{c, n} \quad & u = \gamma \log c + (1 - \gamma) n, \\ \text{s. t.} \quad & w - \tau + zn = c + (m + \beta w) n. \end{aligned}$$

Solving the above problem, we get the optimal fertility as follows:

$$n = \frac{w - \tau}{m + \beta w - z} - \frac{\gamma}{1 - \gamma}, \quad \text{if } w > \frac{\gamma(m - z) + (1 - \gamma)\tau}{1 - \gamma - \beta\gamma}, \quad (9.a)$$

$$c = \frac{\gamma}{1 - \gamma} (m + \beta w - z), \quad \text{if } w > \frac{\gamma(m - z) + (1 - \gamma)\tau}{1 - \gamma - \beta\gamma}, \quad (9.b)$$

$$n = 0, \quad \text{if } w \leq \frac{\gamma(m - z) + (1 - \gamma)\tau}{1 - \gamma - \beta\gamma}, \quad (10.a)$$

$$c = w - \tau, \quad \text{if } w \leq \frac{\gamma(m - z) + (1 - \gamma)\tau}{1 - \gamma - \beta\gamma}. \quad (10.b)$$

Denote  $\underline{w} \equiv \frac{\gamma(m - z) + (1 - \gamma)\tau}{1 - \gamma - \beta\gamma}$ , which is the income level of an individual with a child.

Then, since the income level  $\underline{w}$  decreases as  $z$  increases, an individual can have a child easily by providing child allowance. In contrast, an increase in the tax decreases the income level  $\underline{w}$ .

Therefore, it is harder for an individual to have a child.

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3 Since we assume that an individual is identical, there is no income difference. Therefore, we have the same results as lump-sum tax in the case of proportional income tax.



## 4.2 Government

With regards to the public sector, the government revenue and expenditure are represented by  $\tau$  and  $zn$ , respectively. Therefore, the government budget constraint is expressed by

$$\tau = zn. \quad (11)$$

## 4.3 Equilibrium

To analyze the effects of tax on fertility, we substitute  $\tau$  into equation (9.a) and obtain the following quadratic equations:

$$n^2 (m + \beta w) + \left[ \frac{\gamma}{1 - \gamma} (m + \beta w) - w \right] n - \frac{\gamma}{1 - \gamma} \tau = 0. \quad (12)$$

Calculating the discriminant of the above equation, we have the discriminant as follows:

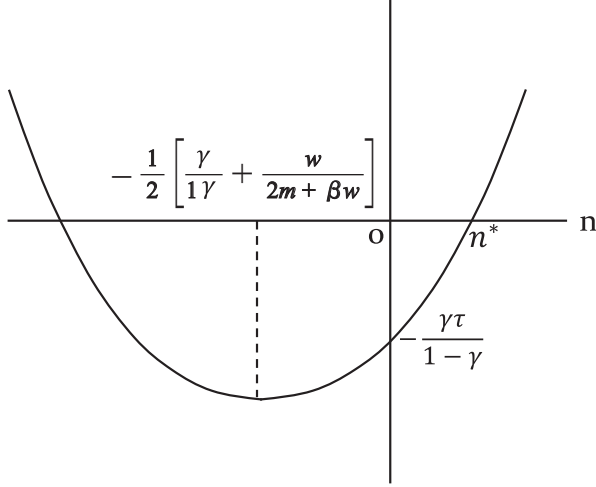
$$D = \left[ \frac{\gamma}{1 - \gamma} (m + \beta w) - w \right]^2 + 4 (m + \beta w) \left( \frac{\gamma}{1 - \gamma} \tau \right) > 0.$$

Hence, we find that equation (12) has two real roots. Solving quadratic equations by using a quadratic formula, we get the following two roots:

$$n^* = \frac{- \left[ \frac{\gamma}{1 - \gamma} (m + \beta w) - w \right] \pm \sqrt{\left[ \frac{\gamma}{1 - \gamma} (m + \beta w) - w \right]^2 + 4 (m + \beta w) \left( \frac{\gamma}{1 - \gamma} \tau \right)}}{2 (m + \beta w)}.$$

Then, we find one root is positive and another is negative as showed Figure 4. Therefore, the optimal fertility is as follows:

$$n^* = \frac{- \left[ \frac{\gamma}{1 - \gamma} (m + \beta w) - w \right] + \sqrt{\left[ \frac{\gamma}{1 - \gamma} (m + \beta w) - w \right]^2 + 4 (m + \beta w) \left( \frac{\gamma}{1 - \gamma} \tau \right)}}{2 (m + \beta w)}. \quad (13)$$



**Figure 4. Solution of quadratic equations**

From equation (13), an increase in tax increases fertility. In the economy, the government ensures a child allowance  $z$  per child. Hence, if the population is sufficiently large, the government must increase the tax on every individual in order to sustain a child allowance level. Even if the government imposes lump-sum tax to ensure child allowance, fertility increases due to the income effect in this case.

### Exercise

In the previous section, we substituted  $z = \frac{\tau}{n}$  into equation (9.a) and considered the effect of tax on fertility. In this exercise, we analyze the effects of child allowance on fertility. We eliminate  $n$  in equation (9.a) and derive fertility in the equilibrium. Substituting equation (11) into equation (9.a), we obtain fertility in the equilibrium.

$$n^* = \frac{1}{m + \beta w} \left[ w - \frac{\gamma}{1 - \gamma} (m + \beta w - z) \right]. \quad (10)$$

Then, from the above equation, an increase in  $z$  raises the fertility.

## 5 Conclusion

This paper presents a static model, in which individual utility is based on quasilinear preferences. In this model, we show that an individual does not always have children. Moreover, we consider the effect of child policy on fertility. We observed that if the government ensures providing a child allowance per child  $z$ , the government could raise fertility in the economy, even if the government imposes increased tax on the household. Moreover, if the government raises the child allowance per child, fertility could increase in the economy.

Finally, it would be interesting to extend this model into a dynamic model and analyze the policy effect on fertility. Moreover, we could introduce the model into a pay-as-you-go pension system and analyze the manner in which the social security system affects fertility in the economy.

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